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Decay of solutions of ordinary differential equations

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Abstract

We investigate the rate of decay of eigenfunctions of Schrödinger equations using a perturbation method which consists of making a perturbation B of the operator L of the form $B[y] = L[y] - (g^{-1}Lg)[y]$, where g is an appropriately chosen function. In our theory we allow B to be either relatively compact or satisfy a certain boundedness condition. We give some examples which apply the results of our main theorems coupled with recent work on the relative boundedness and compactness of differential operators.

1. Introduction

We consider differential expressions of the form

$$L[y] = w^{-1} \sum_{i=1}^{n} p_i(x) y^{(i)}, \tag{1}$$

$-\infty < a \leqslant x < \infty$, and investigate the decay of $\mathcal{L}_w^2(a,\infty)$-solutions of the equation $L[y] = \lambda y$. Conditions on the coefficients $p_i(x)$ are given in Section 2. Asymptotic solutions of differential equations have long been studied using the WKB method and variants of it, e.g., see the book by Eastham [3]. WKB methods

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require a certain smoothness of the coefficients and more importantly require regularity of growth, which rules out many oscillatory functions.

A number of methods have been developed for the decay of eigenfunctions of Schrödinger equations. The reader is referred to Chapter 3 of [6] and the references contained therein for results on Schrödinger equations. Our operators are one-dimensional but may be of arbitrary order. The basic tool is a perturbation method used by Kauffman [8,9] which consists of making a perturbation B of the operator L of the form $B[y] = L[y] - (g^{-1}Lg)[y]$, where g is an appropriately chosen function. In the Kauffman theory B is required to be relatively compact with respect to L. In our theory we allow B to be either relatively compact or satisfy a certain boundedness condition. (See condition (2) of Lemma 1.) This additional condition allows for stronger decay results.

A number of examples are discussed in Section 4, but we illustrate now this stronger decay with the simple example $L[y] = -y'' + q(x)y$ on the interval $a \leqslant x < \infty$ where $q(x)$ is a bounded, real-valued measurable function. If we take $g(x) = e^{\sigma x}$ for $\sigma > 0$ sufficiently small, then the perturbation B,

$$B[y] = L[y] - \big(g^{-1}Lg\big)[y] = \frac{2g'}{g}y' + \frac{g''}{g}y,$$

satisfies our hypothesis which yields that a solution of $L[y] = \lambda y$, $y \in \mathcal{L}^2(a,\infty)$, where $\lambda \in \mathcal{C}$ is not in the essential spectrum, satisfies $\int_a^\infty e^{2\sigma x}|y(x)|^2\,dx < \infty$. By using inequalities for derivatives of sum form, we conclude that $e^{\sigma x}|y(x)|$, $e^{\sigma x}|y'(x)|$, and $e^{\sigma x}|y''(x)|$ approach zero as $x \to \infty$. Now, the perturbation B is not relatively compact, but we produce a relatively compact perturbation if we take $g(x) = e^{\sigma x^\beta}$, $\beta < 1$. For this compact perturbation, our theory gives that y satisfies $\int_a^\infty e^{2\sigma x^\beta}|y(x)|^2\,dx < \infty$, which is a weaker decay of $y(x)$. Note that the order of decay, simple exponential, is maximal as is seen by the example $-y'' = \lambda y$. However, the constant σ produced by our theory is not sharp.

We mention now some of the main features of our theory.

- The coefficients of the operator and the parameter λ may be complex. Thus, the results are not just for self-adjoint problems.
- The order of the differential equation can be arbitrary.
- The decay of the solutions is of the same order as that given by WKB solutions, e.g., solutions of $-y'' + x^2y = 0$ which are $\mathcal{L}^2(0,\infty)$ are shown to decay like $e^{-\sigma x^2}$ for some $\sigma > 0$. Recall the WKB gives decay $x^{-1/2}e^{-x^2/2}$ for the $\mathcal{L}^2(0,\infty)$-solutions. Our theory does not require the smoothness and regularity of growth demanded by WKB theory.
- The decay properties hold for all solutions in the domain of the maximal operator.
- The theory applies to weighted spaces.

The theory of Kauffman has also been used by Mergler and Schultze [10,11] and Schultze [12,13] to study the invariance of deficiency index and essential spectrum under relatively compact perturbations. The class of operators that they consider have polynomial coefficients, but the perturbations are only required to have certain polynomial bounds. The case of invariance of essential spectrum has immediate application to our work since we require $\bar{\lambda}$ not to be in the essential spectrum of L to obtain the decay of a solution of $Ly = \lambda y$. Thus, any knowledge of the location of the essential spectrum is useful in applying our theory. The criteria that Mergler and Schultze develop for relatively compact perturbations of differential operators with polynomial coefficients is quite strong. It is applicable to the theory in this paper in the case that our original operator L has polynomial coefficients and when the perturbations that we consider, which are of the form, $L[y] - (g^{-1}Lg)[y]$, are relatively compact with respect to L.

In Section 2 we state the main hypotheses and define the basic concepts. The general theorems are proved in Section 3. Section 4 contains examples which apply the results of Section 3 when coupled with recent work on the relative boundedness and compactness of differential operators.

2. Preliminaries

We use the following definitions as given by Goldberg [5] and Weidmann [14].

Let X and Y be Banach spaces and let B and L be linear operators, each having domain in X and range in Y. By definition the *graph norm* of L on $D(L)$, denoted $\|\cdot\|_L$, is given by $\|y\|_L = \|y\| + \|Ly\|$.

We say that L is a *closed operator* if its graph is closed, i.e., if for any sequence $y_n \in D(L)$ such that $y_n \to y$ and $Ly_n \to z$, $y \in D(L)$ and $Ly = z$.

We say that B is *relatively bounded with respect to* L (or L*-bounded*) if $D(L) \subseteq D(B)$ and B is bounded on $D(L)$ with respect to $\|\cdot\|_L$, i.e., there exist constants $c, d > 0$ such that $\|By\| \leqslant c\|y\| + d\|Ly\|$ for all $y \in D(L)$. A sequence $\{y_n\}_{n=1}^{\infty}$ is L-bounded if there exists a constant $C > 0$ such that $\|y_n\|_L < C$ for each n. We say that B is *relatively compact with respect to* L (or L*-compact*) if $D(L) \subseteq D(B)$ and B is compact on $D(L)$ with respect to $\|\cdot\|_L$, i.e., if $\{y_n\}_{n=1}^{\infty}$ is a L-bounded sequence, then $\{By_n\}_{n=1}^{\infty}$ contains a convergent subsequence.

The function space setting is the weighted, separable Hilbert space $\mathcal{L}_w^2(J)$, J an interval, which is the space of (equivalence classes of) complex-valued Lebesgue measurable functions y such that $\int_J w|y|^2 < \infty$, where the weight w is a positive Lebesgue measurable function defined on J. If $w \equiv 1$, we denote this space by $\mathcal{L}^2(J)$. We denote the inner product in $\mathcal{L}_w^2(J)$ by $\langle f, g\rangle_w$. For the other Lebesgue spaces we use the notation $\mathcal{L}^p(J)$, $1 \leqslant p \leqslant \infty$.

We consider the differential expression

$$L[y] = w^{-1}\sum_{i=0}^{n} p_i(x)y^{(i)},$$

where each p_i is a complex-valued function on $J = [a, \infty)$ such that $p_i \in \mathcal{L}^{\infty}_{\mathrm{loc}}(J)$, $0 \leqslant i \leqslant n$. The *maximal operator* L_1 is the differential operator defined by L with the largest possible domain in $\mathcal{L}^2_w(J)$ which is mapped into $\mathcal{L}^2_w(J)$, i.e.,

$$D(L_1) = \left\{y \in \mathcal{L}^2_w(J)\colon\ y^{(n-1)} \in AC_{\mathrm{loc}}(J),\ Ly \in \mathcal{L}^2_w(J)\right\},$$

where $AC_{\mathrm{loc}}(J)$ is the set of functions which are absolutely continuous on compact subsets of J. We define the *minimal unclosed operator* L_0' to be the restriction of L_1 to the functions with compact support in the interior of J. The *minimal operator* L_0 is defined to be the closure of L_0'.

We can integrate by parts to obtain an equation of the form $\langle Ly, z\rangle_w = [y,\ z](x) + \langle y, L^+z\rangle_w$, where $[y,\ z](x)$ is called the *Lagrange bilinear form* and L^+ is called the *formal adjoint* of L. We say that L is *formally self adjoint* if $L = L^+$. Note that L_0, L_1, L_0^+ and L_1^+ are closed operators and satisfy $L_0^* = L_1^+$ and $L_1^* = L_0^+$ [4, p. 139]. Recall that for functions y and z with compact support, $\langle L_0 y, z\rangle_w = \langle y, L_0^* z\rangle_w$.

When $L = L^+$ the differential expression above can be written in the form

$$\begin{aligned}\Gamma y(t) = W^{-1}(t)\Bigg\{&\sum_{j=0}^{[n/2]}(-1)^j\big(P_j(t)y^{(j)}(t)\big)^{(j)}\\ &+ i\sum_{j=0}^{[(n-1)/2]}(-1)^j\Big[\big(Q_j(t)y^{(j)}(t)\big)^{(j+1)}\\ &\qquad + \big(Q_j(t)y^{(j+1)}(t)\big)^{(j)}\Big]\Bigg\}, \qquad (2)\end{aligned}$$

where $[\alpha]$ denotes the largest integer less than or equal to α, the complex-valued function y is defined on (a, b), $-\infty \leqslant a < b \leqslant \infty$, the coefficients W, P_j, and Q_j are real-valued functions on (a, b) and $W(t)$ is positive. Each coefficient is assumed to be locally Lebesgue integrable on (a, b). The natural number n is the *order* of the differential expression Γ.

Associated with the formally self-adjoint differential expressions Γ are maximal and minimal operators (Γ_1 and Γ_0, respectively) on the Hilbert space $\mathcal{L}^2_W(a, b)$, which are defined in a similar manner as the maximal and minimal operators associated with the operator L. For example, the maximal operator Γ_1 defined by Γ has domain

$$D(\Gamma_1)=\big\{y\in\mathcal{L}_W^2(a,b)\colon\ y^{\{0\}},y^{\{1\}},\dots,y^{\{n-1\}}\in AC_{\mathrm{loc}}(a,b),\\ \Gamma y\in\mathcal{L}_W^2(a,b)\big\},$$

where each $y^{\{i\}}$, for $i=0,1,\dots,n-1$, is a quasi-derivative [14, pp. 26, 29].

We say that Γ or L is *regular at a* if $a>-\infty$ and the assumptions on the coefficients are satisfied on $[a,b)$ instead of (a,b). We define *regular at b* similarly. If Γ is regular at a and regular at b, then we say that Γ is *regular*. Otherwise, Γ is *singular*.

The *kernel index* of a linear operator A, denoted by $\alpha(A)$, is given by $\alpha(A)=\dim N(A)$, where $N(A)$ is the null space of A. When the range $R(A)$ is closed, we can define the *deficiency index* of A, denoted by $\beta(A)$, by $\beta(A)=\dim R(A)^{\perp}$. (See further explanation of deficiency indices $d_{\pm}$ in [14].) Note that $R(A)$ is closed iff $0\notin\sigma_{\mathrm{ess}}(A)$, where the *essential spectrum* of A is defined by $\sigma_{\mathrm{ess}}(A):=\{\lambda\colon\ R(\lambda I-A)\text{ is not closed}\}$ [14]. If not both $\alpha(A)$ and $\beta(A)$ are infinite, then A has a *Fredholm index* defined as $\kappa(A):=\alpha(A)-\beta(A)$. Note that for any real number r we take $\infty-r=\infty$ and $r-\infty=-\infty$. A closed linear operator with a finite index is called a *Fredholm operator*.

We reference the following result of Brown and Hinton [2, Theorem 2.1].

Theorem 1. *Let $J=[a,\infty)$, $n>0$ and j be integers such that $0\leqslant j\leqslant n-1$, and let N, W, and P be positive measurable functions such that N, W^{-1}, and $P^{-1}\in\mathcal{L}_{\mathrm{loc}}(J)$. Suppose there exists an $\varepsilon_0>0$ and a positive continuous function $f=f(x)$ on J such that*

$$S_1(\varepsilon):=\sup_{x\in J}\left\{f^{2(n-j)}\left[\frac{1}{\varepsilon f}\int_x^{x+\varepsilon f}P^{-1}\right]\left[\frac{1}{\varepsilon f}\int_x^{x+\varepsilon f}N\right]\right\}<\infty$$

and

$$S_2(\varepsilon):=\sup_{x\in J}\left\{f^{-2j}\left[\frac{1}{\varepsilon f}\int_x^{x+\varepsilon f}W^{-1}\right]\left[\frac{1}{\varepsilon f}\int_x^{x+\varepsilon f}N\right]\right\}<\infty$$

for all $\varepsilon\in(0,\varepsilon_0)$. Then there exists a constant $k>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ and $y\in D$,

$$\int_J N\big|y^{(j)}\big|^2\leqslant k\left\{\varepsilon^{-2j}S_2(\varepsilon)\int_J W|y|^2+\varepsilon^{2(n-j)}S_1(\varepsilon)\int_J P\big|y^{(n)}\big|^2\right\}$$

where

$$D=\left\{y\colon\ y\in AC_{\mathrm{loc}}(J),\ \int_J W|y|^2<\infty,\ \textit{and}\ \int_J P\big|y^{(n)}\big|^2<\infty\right\}.$$

3. Main results

We use the following two lemmas in the proof of Theorem 2. Lemma 1 is taken in part from [8, Lemma 2.19].

Lemma 1. *Let F and G be differential expressions of the form* (1) *on the interval (a,b) where F is regular at a, the order of G is less than the order of F, the coefficients of F and G are sufficiently smooth so that $D(F_0') \subseteq D(G_0')$, and F_0 and G_0 satisfy the following two conditions*:

(1) *$R(F_0)$ is closed.*
(2) *Either G_0 is F_0-compact or there exist constants $c, d > 0$ such that $\|Gy\| \leqslant c\|y\| + d\|Fy\|$ for every $y \in D(F_0)$ and $c + d\gamma(F_0) < \gamma(F_0)$, where $\gamma(F_0) = \|F_0^{-1}\|$.*

Then

(i) *$D((F+G)_0) = D(F_0)$ and $(F+G)_0 = F_0 + G_0|_{D(F_0)}$.*
(ii) *$R((F+G)_0)$ is closed.*
(iii) *$\alpha((F^+ + G^+)_1) = \alpha(F_1^+)$, where F^+, G^+ denote the formal adjoints of F, G, respectively, and $(F+G)^+ = F^+ + G^+$.*

Proof. Note that if $R(F_0)$ is closed, then $\|F_0^{-1}\| < \infty$ via the Closed Graph Theorem.

(i) Let $\mathcal{G} = G_0|_{D(F_0)}$. Then Lemma V.3.5 or Corollary V.3.8 of [5, pp. 122–123] implies that $F_0 + \mathcal{G}$ is closed. Since $C_0^\infty \subset D(F_0 + \mathcal{G})$ and $(F+G)_0$ is the minimal closed extension of $(F+G)_0'$, we have $(F+G)_0 \subset F_0 + \mathcal{G}$. Hence,

$$D\big((F+G)_0\big) \subseteq D(F_0 + \mathcal{G}) = D(F_0). \tag{3}$$

Now, let $y \in D(F_0)$. Then there exists a sequence $\{y_n\}_{n=1}^\infty \subset D(F_0')$ such that

$$y_n \to y \quad \text{and} \quad Fy_n \to Fy \quad \text{as } n \to \infty. \tag{4}$$

The F_0-boundedness of G_0 implies that $D(F_0) \subseteq D(G_0)$. Hence, there exists a constant $C > 0$ such that $\|Gy_n - Gy\| \leqslant C(\|y_n - y\| + \|Fy_n - Fy\|)$. Via (4) and the above inequality, we have $Gy_n \to Gy$ as $n \to \infty$. Therefore the sequence $\{(F+G)y_n\}_{n=1}^\infty$ converges to $(F+G)y$. By definition, $y \in D((F+G)_0)$. Thus,

$$D(F_0) \subseteq D\big((F+G)_0\big). \tag{5}$$

Via (3) and (5) we have that (i) holds.

(ii) Since F_0 is a minimal operator with a regular endpoint a, we know that $\alpha(F_0) = 0$ ([14, Theorem 3.12] and [5, Lemma VI.2.4]). Thus, F_0 has an index. Since $R(F_0)$ is closed, Theorem V.3.6(i) or Corollary V.3.8 of [5, pp. 122–123] implies that $R((F+G)_0)$ is closed.

(iii) Since $R(F_0)^\perp = N(F_0^*)$ [7, IV.5.16] and $F_0^* = F_1^+$, $R(F_0)^\perp = N(F_1^+)$. Hence,

$$\beta(F_0) = \dim R(F_0)^\perp = \dim N\big(F_0^*\big) = \dim N\big(F_1^+\big) = \alpha\big(F_1^+\big), \tag{6}$$

and similarly,

$$\beta\big((F+G)_0\big) = \alpha\big((F+G)_1^+\big). \tag{7}$$

Theorem V.3.6(ii) or Corollary V.3.8 of [5, pp. 122–123] implies that $\kappa(F_0) = \kappa((F+G)_0)$; and since F_0 and $(F+G)_0$ are minimal operators with a regular endpoint a, we conclude that $\alpha(F_0) = \alpha((F+G)_0) = 0$ and hence

$$\beta(F_0) = \beta\big((F+G)_0\big). \tag{8}$$

Equations (6), (7) and (8) imply (iii) holds. $\square$

Lemma 2. *Let F be a differential expression of the form* (1) *on the interval J with smooth coefficients. Let $G = g^{-1}Fg - F$, where g is a positive, n times continuously differentiable function on J and $g^{-1} = 1/g$. Then the order of G is less than the order of F and $(g^{-1}Fg)^+ = gF^+g^{-1}$.*

Proof. A long calculation using the product rule for differentiation shows that the highest order derivatives of F and $g^{-1}Fg$ cancel, thus the order of G is less than the order of F.

Since the adjoint is found by successive integration by parts, we have for $y, z \in \mathcal{L}_w^2(J)$ of class C^n with compact support in the interior of J,

$$\begin{aligned}\langle g^{-1}Fgy, z\rangle_w &= \int_J \overline{z(x)}\,w(x)\big(g^{-1}Fgy\big)(x)\,dx \\ &= \int_J \overline{\big(g^{-1}z\big)(x)}\,w(x)(Fgy)(x)\,dx \\ &= \int_J \overline{\big(F^+g^{-1}z\big)(x)}\,w(x)(gy)(x)\,dx \\ &= \int_J \overline{\big(gF^+g^{-1}z\big)(x)}\,w(x)y(x)\,dx \\ &= \langle y, gF^+g^{-1}z\rangle_w.\end{aligned}$$

Therefore, $(g^{-1}Fg)^+ = gF^+g^{-1}$. $\square$

Theorem 2. *Let I be the identity operator and let F and G be as in Lemma* 2. *Suppose $(F - \bar{\lambda}I)_0$ and G_0 satisfy conditions* (1) *and* (2) *of Lemma* 1. *If $g \geqslant \alpha$ on J for some $\alpha > 0$, then the map $y \mapsto g^{-1}y$ is one-to-one from $N((g(F^+ - \lambda I) \times g^{-1})_1)$ onto $N((F^+ - \lambda I)_1)$.*

Proof. Since g^{-1} is bounded above on J, $\int_J w(x)|y(x)|^2\,dx < \infty$ implies that $\int_J w(x)g^{-2}(x)|y(x)|^2\,dx < \infty$. Moreover, if we let $g(F^+ - \lambda I)g^{-1}y = 0$ for $y \in \mathcal{L}_w^2(J)$, then $(F^+ - \lambda I)g^{-1}y = 0$. Therefore, $g^{-1}y \in N((F^+ - \lambda I)_1)$. Hence, the map $y \mapsto g^{-1}y$ maps $N((g(F^+ - \lambda I)g^{-1})_1)$ into $N((F^+ - \lambda I)_1)$ and is one-to-one since g is positive. By Lemma 2

$$G^+ = \left(g^{-1}Fg - F\right)^+ = \left(g^{-1}Fg\right)^+ - F^+ = gF^+g^{-1} - F^+.$$

Using this expression for G^+ and $(F - \bar{\lambda}I)^+ = F^+ - \lambda I$, we have via Lemma 1(iii) that

$$\begin{aligned}\alpha\big((F^+ - \lambda I)_1\big) &= \alpha\big((F - \bar{\lambda}I)_1^+\big) = \alpha\big(\big((F - \bar{\lambda}I)^+ + G^+\big)_1\big)\\ &= \alpha\big((F^+ - \lambda I + G^+)_1\big) = \alpha\big(\big(gF^+g^{-1} - \lambda I\big)_1\big)\\ &= \alpha\big(\big(g(F^+ - \lambda I)g^{-1}\big)_1\big).\end{aligned}$$

Since the null spaces have the same finite dimension and the map $y \mapsto g^{-1}y$ is one-to-one, the map is onto. □

Remark. We conclude from Theorem 2 that the inverse map $z \mapsto gz$ is one-to-one from $N((F^+ - \lambda I)_1)$ onto $N((g(F^+ - \lambda I)g^{-1})_1)$. Therefore, $z \in N((F^+ - \lambda I)_1)$ implies that $gz \in N((g(F^+ - \lambda I)g^{-1})_1)$, i.e., $\int_J w(x)g^2(x)|z(x)|^2\,dx < \infty$.

4. Applications

Now, we examine several examples to obtain upper bounds on the rate of eigenfunction decay. In each example the given differential expression L is formally self-adjoint, i.e., $L = L^+$. If $L = L^+$ and λ is nonreal, then $\lambda \notin \sigma_{\mathrm{ess}}(L_0)$. Therefore, for all complex-valued λ, $\bar{\lambda} \notin \sigma_{\mathrm{ess}}(L_0)$ is equivalent to $\lambda \notin \sigma_{\mathrm{ess}}(L_0)$. We use this fact implicitly in applying Theorem 2. In all three examples below, we take the weight function $w \equiv 1$.

Example 1. Suppose p and q are real-valued functions satisfying $p \in AC_{\mathrm{loc}}(J)$, $J = [a, \infty)$, $a > 0$, $q \in \mathcal{L}^\infty(J)$, $p > 0$ and $|p'(x)| \leqslant K\sqrt{p(x)}$ a.e. on J for some positive constant K. Let L_1 (L_0) be the maximal (minimal) operator associated with the differential expression $L[y] = -(p(x)y')' + q(x)y$, and let B_1 (B_0) be the maximal (minimal) operator associated with the differential expression $B[y] = L[y] - (g^{-1}Lg)[y]$ with g defined by

$$g(x) = \exp\left[\sigma \int_a^x \frac{1}{\sqrt{p(s)}}\,ds\right].$$

After some calculations

$$B[y] = B^{[1]}[y] + B^{[0]}[y],$$

where

$$B^{[1]}[y] = 2\sigma\sqrt{p(x)}\,y' \quad \text{and} \quad B^{[0]}[y] = \left(\sigma^2 + \frac{\sigma p'(x)}{2\sqrt{p(x)}}\right)y.$$

The proof of Theorem 2.1 of Anderson and Hinton [1] shows that $B_0^{[j]}$ is relatively bounded with respect to $L_0 - \bar{\lambda} I$ and the constants c_j and d_j in $\|B^{[j]}y\| \leqslant c_j\|y\| + d_j\|(L - \bar{\lambda} I)y\|$, $y \in D(L_0)$, are bounded by a constant (independent of ε and y) times the quantities $\varepsilon^{-2j}S_2(\varepsilon), \varepsilon^{2(2-j)}S_1(\varepsilon)$, $j = 0, 1$. Hence, $B_0[y] = B_0^{[1]}[y] + B_0^{[0]}[y]$ has the bound $\|By\| \leqslant (c_0 + c_1)\|y\| + (d_0 + d_1)\|(L - \bar{\lambda} I)y\|$, $y \in D(L_0)$. It is further shown in the proof of Theorem 2.1 of [1] that the application of our Theorem 1 with $n = 2$, $W(x) = 1$, $P(x) = p(x)^2$, $f(x) = \sqrt{p(x)}$, and $N(x) = (\sigma^2 + \sigma p'(x)/\sqrt{p(x)}\,)^2$ for $j = 0$ and $N(x) = (2\sigma\sqrt{p(x)}\,)^2$ for $j = 1$ yields that the quantities $S_1(\varepsilon)$ and $S_2(\varepsilon)$ are proportional to

$$\sup_{x\in J} \frac{1}{\varepsilon\sqrt{p(x)}} \int_x^{x+\varepsilon\sqrt{p(x)}} \left|\sigma^2 + \frac{\sigma p'(s)}{2\sqrt{p(s)}}\right|^2 ds$$

in the case $j = 0$, and in the case $j = 1$

$$\sup_{x\in J} \frac{1}{\varepsilon\sqrt{p(x)}} \int_x^{x+\varepsilon\sqrt{p(x)}} p(s)^{-1}\left|2\sigma\sqrt{p(s)}\right|^2 ds.$$

Notice that the first expression is bounded above by a constant multiple of σ^2 because of the condition on $|p'(x)|$, and the second expression is equivalent to $4\sigma^2$. Hence, we can choose $\sigma > 0$ sufficiently small so that there exist constants $c, d > 0$ such that $\|By\| \leqslant c\|y\| + d\|(L - \bar{\lambda} I)y\|$ for every $y \in D(L_0)$ and

$$c + d\gamma(L_0 - \bar{\lambda} I) < \gamma(L_0 - \bar{\lambda} I). \tag{9}$$

Therefore, L and B satisfy the hypotheses of Theorem 2 for $\bar{\lambda} \notin \sigma_{\mathrm{ess}}(L_0)$. We apply the remark following Theorem 2 to conclude that if $y \in N((L - \lambda I)_1)$, then

$$\int_J \exp\left[2\sigma \int_a^x \frac{1}{\sqrt{p(s)}}\,ds\right] \left|y(x)\right|^2 dx < \infty. \tag{10}$$

Hence, every solution $y \in D(L_1)$ of $Ly = \lambda y$, $\lambda \notin \sigma_{\mathrm{ess}}(L_1)$, decays exponentially as in (10).

Notice that if we let $p(x) \equiv 1$ on J, we obtain $\int_a^\infty e^{2\sigma x}|y(x)|^2\,dx < \infty$. Since $-y'' + q(x)y = \lambda y$ and $q(x) \in \mathcal{L}^\infty(J)$, it follows that $\int_a^\infty e^{2\sigma x}|y''(x)|^2\,dx < \infty$.

A standard interpolation inequality (see Lemma 2.1 of [2]) gives that there is a constant k so that if $[c, c+1] \subset J$, then for $x \in [c, c+1]$

$$\left|f^{(j)}(x)\right| \leqslant k\left(\int_{c}^{c+1}\left|f(x)\right| dx + \int_{c}^{c+1}\left|f''(x)\right| dx\right), \quad j = 0, 1, \tag{11}$$

for all f such that $f, f' \in AC_{\mathrm{loc}}(J)$. Using $f = y$ in (11) and applying the Cauchy–Schwarz inequality gives for $x \in [c, c+1]$,

$$\begin{aligned}\left|y^{(j)}(x)\right| \leqslant k&\left\{\left(\int_{c}^{c+1} e^{-2\sigma x}\, dx\right)^{1/2}\right\}\\ &\times\left\{\left(\int_{c}^{c+1} e^{2\sigma x}\left|y(x)\right|^2 dx\right)^{1/2} + \left(\int_{c}^{c+1} e^{2\sigma x}\left|y''(x)\right|^2 dx\right)^{1/2}\right\}\end{aligned}$$

from which it follows that $e^{\sigma x}y(x)$, $e^{\sigma x}y'(x) \to 0$ as $x \to \infty$; hence $e^{\sigma x}y''(x) \to 0$ as $x \to \infty$ since $q \in \mathcal{L}^{\infty}(J)$. This gives the first result mentioned in the Introduction.

Example 2. Suppose $q \in AC_{\mathrm{loc}}(J)$, $J = [a, \infty)$, $a > 0$, is a real-valued function satisfying $q \geqslant K_1$ and $|q'(x)| \leqslant K_2 q(x)^{3/2}$ for $x \in J$ and some positive constants K_1 and K_2. Let L_1 (L_0) be the maximal (minimal) operator associated with the differential expression $L[y] = -y'' + q(x)y$, and let B_1 (B_0) be the maximal (minimal) operator associated with the differential expression $B[y] = L[y] - (g^{-1}Lg)[y]$ with g defined by

$$g(x) = \exp\left[\sigma \int_{a}^{x} \sqrt{q(s)}\, ds\right].$$

After some calculations $B[y] = B^{[1]}[y] + B^{[0]}[y]$, where

$$B^{[1]}[y] = 2\sigma\sqrt{q(x)}\,y' \quad \text{and} \quad B^{[0]}[y] = \left(\sigma^2 q(x) + \frac{\sigma q'(x)}{2\sqrt{q(x)}}\right)y.$$

The proof of Theorem 3.1 of Anderson and Hinton [1] shows that $\|B^{[j]}y\| \leqslant d_j\|Ly\|$, $y \in D(L_0)$, where the constant d_j (independent of ε and y) is a linear combination the quantities $\varepsilon^{2(2-j)}S_1(\varepsilon)$ and $\varepsilon^{-2j}S_2(\varepsilon)$, $j = 0, 1$. Hence, $B_0[y] = B_0^{[1]}[y] + B_0^{[0]}[y]$ has the bound

$$\|By\| \leqslant (d_0 + d_1)|\lambda|\|y\| + (d_0 + d_1)\left\|(L - \bar{\lambda}I)y\right\|, \quad y \in D(L_0).$$

It is further shown in the proof of Theorem 3.1 of [1] that the application of our Theorem 1 with $n = 2$,

$$W(x) = q(x)^2, \qquad P(x) = 1, \qquad f(x) = \frac{1}{\sqrt{q(x)}}, \quad \text{and}$$
$$N(x) = \left(\sigma^2 q(x) + \frac{\sigma q'(x)}{2\sqrt{q(x)}}\right)^2 \quad \text{for } j = 0 \quad \text{and}$$
$$N(x) = \left(2\sigma\sqrt{q(x)}\right)^2 \quad \text{for } j = 1$$

yields that the quantities $S_1(\varepsilon)$ and $S_2(\varepsilon)$ are proportional to

$$\sup_{x\in J} \frac{\sqrt{q(x)}}{\varepsilon} \int_x^{x+\varepsilon/\sqrt{q(x)}} q(s)^{-2} \left|\sigma^2 q(s) + \frac{\sigma q'(s)}{2\sqrt{q(s)}}\right|^2 ds$$

in the case $j = 0$, and in the case $j = 1$

$$\sup_{x\in J} \frac{\sqrt{q(x)}}{\varepsilon} \int_x^{x+\varepsilon/\sqrt{q(x)}} q(s)^{-1} \left|2\sigma\sqrt{q(s)}\right|^2 ds.$$

Notice that the first expression is bounded above by a constant multiple of σ^2 because of the condition on $|q'(x)|$, and the second expression is equivalent to $4\sigma^2$. As in Example 1, we can choose $\sigma > 0$ sufficiently small so that there exist constants $c, d > 0$ such that $\|By\| \leqslant c\|y\| + d\|(L - \bar{\lambda}I)y\|$ for every $y \in D(L_0)$ and inequality (9) holds. Therefore, L and B satisfy the hypotheses of Theorem 2 for $\bar{\lambda} \notin \sigma_{\text{ess}}(L_0)$. We apply the remark following Theorem 2 to conclude that if $y \in N((L - \lambda I)_1)$, then

$$\int_J \exp\left[2\sigma \int_a^x \sqrt{q(s)}\, ds\right] \left|y(x)\right|^2 dx < \infty. \tag{12}$$

Hence, every solution $y \in D(L_1)$ of $Ly = \lambda y$, $\lambda \notin \sigma_{\text{ess}}(L_1)$, decays exponentially as in (12).

Notice that if we let $q(x) = x^2$ on J, we obtain the second result mentioned in the Introduction.

Example 3. Suppose $q \in \mathcal{L}_{\text{loc}}(J)$, $J = [a, \infty)$, $a > 0$, is a real-valued function satisfying $|q(x)| \leqslant Kx^{\gamma-2}$ for $x \in J$, $\gamma \geqslant 2$ and some sufficiently small positive constant K. Let L_1 (L_0) be the maximal (minimal) operator associated with the differential expression $L[y] = -(x^\gamma y')' + q(x)y$, and let B_1 (B_0) be the maximal (minimal) operator associated with the differential expression $B[y] = L[y] - (g^{-1}Lg)[y]$ with g defined by $g(x) = \sigma x^\sigma$ for sufficiently small $\sigma > 0$. After some calculations $B[y] = B^{[1]}[y] + B^{[0]}[y]$, where $B^{[1]}[y] = 2\sigma x^{\gamma-1}y'$ and $B^{[0]}[y] = \sigma(\sigma + \gamma - 1)x^{\gamma-2}y$. The proof of Theorem 4.1 of Anderson and Hinton [1] shows that $\|B^{[j]}y\| \leqslant d_j\|Ly\|$, $y \in D(L_0)$, where the constant d_j

(independent of ε and y) is a linear combination the quantities $\varepsilon^{2(2-j)}S_1(\varepsilon)$ and $\varepsilon^{-2j}S_2(\varepsilon)$, $j = 0, 1$. Hence, $B_0[y] = B_0^{[1]}[y] + B_0^{[0]}[y]$ has the bound

$$\|By\| \leqslant (d_0 + d_1)|\lambda|\|y\| + (d_0 + d_1)\left\|(L - \bar{\lambda}I)y\right\|, \quad y \in D(L_0).$$

It is further shown in the proof of Theorem 4.1 of [1] that the application of our Theorem 1 with $n = 2$, $W(x) = x^{2\gamma-4}$, $P(x) = x^{2\gamma}$, $f(x) = x$, and $N(x) = (\sigma(\sigma+\gamma-1)x^{\gamma-2})^2$ for $j = 0$ and $N(x) = (2\sigma x^{\gamma-1})^2$ for $j = 1$ yields that the quantities $S_1(\varepsilon)$ and $S_2(\varepsilon)$ are proportional to

$$\sup_{x\in J} \frac{1}{\varepsilon x} \int_x^{x+\varepsilon x} s^{2(2-\gamma)}\left|\sigma(\sigma+\gamma-1)s^{\gamma-2}\right|^2 ds$$

in the case $j = 0$, and in the case $j = 1$

$$\sup_{x\in J} \frac{1}{\varepsilon x} \int_x^{x+\varepsilon x} s^{2(1-\gamma)}\left|2\sigma s^{\gamma-1}\right|^2 ds.$$

Notice that each expression is a constant multiple of σ^2. As in Example 1, we can choose $\sigma > 0$ sufficiently small so that there exist constants $c, d > 0$ such that $\|By\| \leqslant c\|y\| + d\|(L - \bar{\lambda}I)y\|$ for every $y \in D(L_0)$ and inequality (9) holds. Therefore, L and B satisfy the hypotheses of Theorem 2 for $\bar{\lambda} \notin \sigma_{\text{ess}}(L_0)$. We apply the remark following Theorem 2 to conclude that if $y \in N((L - \lambda I)_1)$, then

$$\int_J x^{2\sigma}\left|y(x)\right|^2 dx < \infty. \tag{13}$$

Hence, every solution $y \in D(L_1)$ of $Ly = \lambda y$, $\lambda \notin \sigma_{\text{ess}}(L_1)$, decays as in (13).

Note that we only get polynomial decay in Example 3. This is to expected for if $\gamma = 2$ and $q(x) = 0$, then the essential spectrum is $[1/4, \infty)$; and for $\lambda < 1/4$ the $\mathcal{L}^2(J)$-solutions of $-(x^2y')' = \lambda y$ are multiples of the function x^r, $r = (-1 - (1 - 4\lambda)^{1/2})/2$.

We remark that Lemmas 1 and 2 and Theorem 2 apply to the more general formal differential expressions of the form (2) in which the coefficients W, P_j, and Q_j are $(m \times m)$-matrix valued functions on (a, b) and $W(t)$ is positive definite.

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References

[1] T.G. Anderson, D.B. Hinton, Relative boundedness and compactness theory for second-order differential operators, J. Inequal. Appl. 1 (1997) 375–400.

[2] R.C. Brown, D.B. Hinton, Sufficient conditions for weighted inequalities of sum form, J. Math. Anal. Appl. 112 (1985) 563–578.

[3] M.S.P. Eastham, The Asymptotic Solution of Linear Differential Systems, Clarendon, Oxford, 1989.

[4] H. Frentzen, Equivalence, adjoints, and symmetry of quasi-differential expressions with matrix-valued coefficients and polynomials in them, Proc. Roy. Soc. Edinburgh Sect. A 92 (1982) 123–146.

[5] S. Goldberg, Unbounded Linear Operators: Theory and Applications, Dover, New York, 1985.

[6] P.D. Hislop, I.M. Sigal, Introduction to Spectral Theory with Applications to Schrödinger Operators, Springer-Verlag, New York, 1996.

[7] T. Kato, Perturbation Theory for Linear Operators, Springer-Verlag, New York, 1980.

[8] R.M. Kauffman, On the limit-n classification of ordinary differential operators with positive coefficients, Proc. London Math. Soc. (3) 35 (1977) 496–526.

[9] R.M. Kauffman, On the limit-n classification of ordinary differential operators with positive coefficients II, Proc. London Math. Soc. (3) 41 (1980) 499–515.

[10] B. Mergler, B. Schultze, A perturbation method and the limit-point case of even order symmetric differential expressions, Proc. Roy. Soc. Edinburgh Sect. A 94 (1983) 121–135.

[11] B. Mergler, B. Schultze, On the stability of the limit-point property of “Kauffman expressions”, Proc. Roy. Soc. Edinburgh Sect. A 103 (1986) 73–89.

[12] B. Schultze, Odd-order differential expressions with positive supporting coefficients, Proc. Roy. Soc. Edinburgh Sect. A 105 (1987) 167–192.

[13] B. Schultze, Spectral properties of not necessarily self-adjoint linear differential operators, Adv. Math. 83 (1990) 75–95.

[14] J. Weidmann, Spectral Theory of Ordinary Differential Operators, in: Lecture Notes in Math., Vol. 1258, Springer-Verlag, New York, 1987.